Algebraic-geometric aspects of function field analogues to abelian varieties

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This communication is a continuation of [6, 7, 8, 9].

Let p be a prime number, $q = p^n$, $\mathbb{F}_q$ be the field with q elements and characteristic p, $\mathbb{F}$ be a finite field extension of a finite field $\mathbb{F}_q$.
We extend the case of algebraic number fields [6, 7] to the case of function fields in characteristic $p > 0$ and construct function field analogues to abelian varieties of elliptic and hyperelliptic curves appeared in [8, 9]. In the last case we investigate function field analogues to abelian varieties which are Jacobian varieties of hyperelliptic curves in characteristic $p > 0$. Recall that for hyperelliptic curves the function field analogues to abelian varieties are function field analogues to Jacobian varieties of the curves. For Jacobians it is possible to define corresponding $p-$divisible groups. We plan to present results on function field analogues to p-divisible groups of the Jacobian varieties.

Moduli and estimates for hyperelliptic curves of genus $g \geq 2$ over $\mathbb{F}_p$.

Let

$$C : y^2 = f(x)$$

be an algebraic curve and let $Disk(C)$ be the discriminant of $f(x)$. Consider hyperelliptic curve of genus $g \geq 2$ over prime finite field $\mathbf{F}_p$

$$C_g : y^2 = f(x),\ D(f) \neq 0.$$

For projective closure of C_g the quasiprojective variety

$$S_{g,p} = \{\mathbf{P}^{2g+2}(\mathbf{F}_p) \setminus (Disk(C_g) = 0)\}$$

parametrizes all hyperelliptic curves of genus g over $\mathbf{F}_p$. By well known Weil bound (affine case)

$$|\#C_g(\mathbf{F}_p) - p| \leq 2g\sqrt{p}.$$

where $\#C$ is the number of points on the curve C over ground field. As we can see from Weil (and some more strong) bounds, for $p \geq 17$ any hyperelliptic curve of genus $g = 2$ has points in $\mathbb{F}_p$ for these prime p. Also for $g = 3$ every hyperelliptic (h) curve of genus 3 has points in $\mathbb{F}_p$ for $p \geq 37$. For $p = 2, 3, 5, 7, 11$ there are examples of h-curves of genus 2 that have not points in $\mathbb{F}_p$. By author's computations any h-curve of genus 2 over $\mathbb{F}_{13}$ has points in the field. Similarly, for $p = 2, 3, 5, 7, 11, 13, 17$ there are examples of h-curves of genus 3 that have not points in $\mathbb{F}_p$.

Theorem 1. [8]. *Let $p \equiv 3 \bmod 4$. Under $p \geq 11$ there is such $a \in \mathbb{F}_p$ that the equation*

$$y^2 = x^{\frac{p-1}{2}} + a$$

has no solutions in $\mathbb{F}_p$.

Global $\mathfrak{G}$-shtukas and local $\mathbb{P}$-shtukas [1, 2, 3, 4].

Definition 2. (Hartl, Rad [1, 2]) Let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$. A global $\mathfrak{G}$-shtuka $\overline{\mathcal{G}}$ over an $\mathbb{F}_q$-scheme S is a tuple $(\mathcal{G}, s_1, \ldots, s_n, \tau)$ consisting of a $\mathfrak{G}$-torsor $\mathcal{G}$ over $C_S := C \times_{\mathbb{F}_q} S$, an n-tuple of (characteristic) sections $(s_1, \ldots, s_n) \in C^n(S)$ and a Frobenius connection τ defined outside the graphs of the sections s_i.

For Jacobian varieties it is possible to define corresponding p-divisible groups and their function field analogues.

Definition 3. (Hartl, Rad [1]) Let $\mathbb{P}$ be a flat affine group scheme of finite type over $Spec\,\mathbb{F}[[z]]$ and $\mathfrak{G}$ is a flat affine group scheme of finite type over a smooth projective geometrically irreducible curve over $\mathbb{F}_q$

Recall that local $\mathbb{P}$-shtukas are the functional field analogs of p-divisible groups with additional structure and moduli stacks of global $\mathfrak{G}$-shtukas are the functional field analogs for Shimura varieties. In some cases $\mathbb{P}$ is a paraholic Bruhat-Tits group scheme by Pappas, Rapoport [5] and $\mathfrak{G}$ is a parahoric Bruhat-Tits group scheme over a smooth projective curve over finite field $\mathbb{F}_q$ with q elements of characteristic p. Investigations by U. Hartl [3], by Hartl, Arasteh Rad [1, 2], by U. Hartl, E. Viehmann [4] continue works of V. G. Drinfeld , L. Lafforgue, G. Faltings.

If will sufficient time we plan to give a short review of history of these research.

References

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